The Stochastic Logarithmic Norm for Stability Analysis of Stochastic Differential Equations

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Abstract

To analyze the stability of Itô stochastic differential equations with multiplicative noise, we introduce the stochastic logarithmic norm. The logarithmic norm was originally introduced by G. Dahlquist in 1958 as a tool to study the growth of solutions to ordinary differential equations and for estimating the error growth in discretization methods for their approximate solutions. We extend the concept to the stability analysis of Itô stochastic differential equations with multiplicative noise. Stability estimates for linear Itô SDEs using the one, two and ∞-norms in the l-th mean, where $1 \leq l < \infty$, are derived and the application of the stochastic logarithmic norm is illustrated with examples.

Key words: Logarithmic norms, Stochastic differential equations.

1. Introduction

For $A \in \mathbb{C}^{n\times n}$ and $X(t) \in \mathbb{C}^n$, we consider the ordinary differential equation (ODE) $dX_t = AX_t dt, X(0) = x_0$. Then we have $\|X(t)\| \leq \|x_0\| e^{\mu(A)t}$ where $\mu(A)$ is the logarithmic norm of the matrix A [4,19,20]. If $\mu(A) < 0$, then the ODE is asymptotically stable. Also, $\mu(A)$ using the matrix 2-norm gives an estimate [13] for the pseudospectrum [15] of A: $\max \Re\lambda_\epsilon(A) - \epsilon \leq \mu(A)$ where $1 \gg \epsilon > 0$. Since the pseudospectrum captures the

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stability of the numerical solution of the ODE over a finite number of time steps under the effect of local stiffness and nonnormality of A (we shall refer to this as the numerical stability) [6,7], having $\mu(A) < 0$ implies numerical stability in addition to the asymptotic stability of the ODE. As is already shown in [6], transient numerical stability affects the computation and choice of methods for numerical integration of the ODE.

In this paper we extend the classical logarithmic norm to the stability analysis of Itô stochastic differential equations (SDE) and introduce the stochastic logarithmic norm for estimating the numerical stability of an SDE in order to facilitate the selection of stiff and balanced stochastic numerical integration schemes. Letting $A, B \in \mathbb{C}^{n\times n}, X(t) \in \mathbb{C}^n,\ B \in \mathbb{C}^{n\times n}$, the Itô SDE with a single channel of multiplicative noise is considered in the form of $dX_t = AX_t dt + BX_t dW_t$ given the initial condition that $X(0) = x_0$ with probability (w.p.) 1, where W is the one dimensional Wiener process such that $\int_0^s dW_t \sim N(0, s)$, i.e., is standard Gaussian distributed with mean 0 and variance s. In our definition of the stochastic logarithmic norm we shall use the the matrix p-norm induced by the vector p-norm and the expectation of the l-th raw moment, i.e., the l-th mean of the solution to the linear multiplicative SDE. The stochastic logarithmic norm is computed over the sample paths as the expected logarithmic norm of the system in the sense of the existence of a generalized derivative of the Wiener process which itself is not obligatory differentiable with respect to time.

Throughout the paper the following standard assumptions are made as in [9]. Let there be a common probability space $(\Omega, \mathcal{A}, P)$ with index $t \in \mathcal{T} \subset \mathbb{R}$ on which the stochastic process $X(t)$ is a collection of random variables. The Wiener process $W = \{W_t, t \geq t_0\}$ is associated with an increasing family of σ-algebras $\{\mathcal{A}_t, t \geq t_0\}$. For the general case of multi-dimensional noise, each component of $\{W_t^{(i)}\}$ is $\mathcal{A}_t$-measurable with $\mathbf{E}(W(t_0)) = 0$ w.p. 1, $\mathbf{E}(W(t)|\mathcal{A}_{t_0}) = 0$, $\mathbf{E}((W_t^{(i)} - W_s^{(i)})(W_t^{(j)} - W_s^{(j)})|\mathcal{A}_s) = \delta_{i,j}(t-s)$ for $t_0 \leq s \leq t$ and $\Delta W = W(t_{n+1}) - W(t_n)$, the component wise increments of the multi-dimensional Wiener process, are independent of each other at all points in the partition of the time interval $\mathcal{T}$: $t_0 \leq t_1 \leq t_2 \ldots \leq t_r \leq t_{r+1} \leq \ldots \leq t_N = t_f$. The initial value X_0 is assumed to be $\mathcal{A}_{t_0}$-measurable with $\|X_0\|_p < \infty$ w.p. 1. All expectations on a function $\phi(X_t)$ are evaluated as $\mathbf{E}(\phi(X_t)|\mathcal{A}_t)$ unless otherwise stated. Inequalities and equalities involving random variables hold almost surely where applicable.

The stability analysis in [17] uses test equations with scalar and 2-by-2 matrix coefficients having multiplicative noise of dimension one [16]. Some of the analysis uses the classical logarithmic norm to establish the stability of the moment equations (derived from the SDE) which are deterministic. The present approach with stochastic logarithmic norm generalizes the stability analysis of SDEs as found in [16,17].

1.1. *Classical Logarithmic Norm*

For $1 \leq p \leq \infty$, the p-norm on $\mathbb{C}^n$ is given by

$$\|x\|_p := \begin{cases} \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}, & \text{for } 1 \leq p < \infty, \\ \max_{1\leq j\leq n} |x_j|, & \text{for } p = \infty. \end{cases}$$

Obviously, $\|x\|_2 := (x^H x)^{1/2}$ is the 2-norm on $\mathbb{C}^n$. For $A \in \mathbb{C}^{n\times n}$, the spectrum $\Lambda(A)$ of A is given by $\Lambda(A) := \{\lambda \in \mathbb{C} : \mathbf{rank}(A - \lambda I) < n\}$. We denote a matrix p-norm on $\mathbb{C}^{n\times n}$ induced by the vector p-norm as $\|\cdot\|_p$ for $p = 1, 2, \infty$ and define these norms as $\|A\|_2 := \max_j\{\sqrt{\lambda_j} : \lambda_j \in \Lambda(AA^H)\}$, $\|A\|_1 := \max_{j=1,\dots,n}\left(\sum_{i=1}^n |a_{ij}|\right)$ and $\|A\|_\infty := \max_{i=1,\dots,n}\left(\sum_{j=1}^n |a_{ij}|\right)$. Then the logarithmic norm for a single matrix is defined as

$$\mu_p(A) := \lim_{h\to 0^+} \frac{\|I + hA\|_p - 1}{h}.$$

For the $1, 2$ and ∞-norms, the classical logarithmic norms, respectively, are computed ([5],Vol 1) as $\mu_1(A) := \max_j \left(\Re(a_{jj}) + \sum_{j\neq i}^n |a_{ij}|\right)$, $\mu_2(A) := \dfrac{\lambda_{\max}(A + A^H)}{2}$ and as $\mu_\infty(A) := \max_i \left(\Re(a_{ii}) + \sum_{i\neq j}^m |a_{ij}|\right)$.

1.2. *Stability of the SDE*

The stability of the vector SDE with single channel multiplicative noise is defined as follows.

Definition 1 *[2,3] The equilibrium solution $X_t \equiv 0$ to $dX_t = AX_t dt + BX_t dW_t$ is stochastically stable in the l-th mean (l is a finite integer ≥ 1) using a vector p-norm if $\forall \epsilon > 0$, $\exists \delta > 0$ such that*

$$\mathbf{E}(\|X(t)\|_p^l) < \epsilon \quad \forall\, t \geq t_0 \quad \textit{and} \quad \|X(t_0)\|_p < \delta \;\textit{w.p. } 1 \tag{1}$$

and is asymptotically stable in l-th mean if in addition, $\exists \delta_0 > 0$ such that

$$\lim_{t\to\infty} \mathbf{E}(\|X_t\|_p^l) = 0 \quad \forall\, \|X(t_0)\|_p < \delta_0 \;\textit{w.p. } 1. \tag{2}$$

2. Background

In the following items we review the existing stability analysis of linear stochastic differential equations with multiplicative noise.

(a) In [16] scalar stochastic differential equations of the form $dX_t = \lambda X_t dt + \beta X_t dW_t$ with $X_0 = 1$ w.p. 1, where λ and β are constants have been considered and it is shown that the above stochastic differential equation is mean square stable using the 2-norm if $2\Re(\lambda) + |\beta|^2 \leq 0$ when λ, β are complex scalars.

(b) In [17] vector stochastic differential equations, with single channel multiplicative noise, of the form $dX_t = DX_t dt + BX_t dW_t$, where $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and $B = \begin{pmatrix} \alpha_1 & \beta_1 \\ \beta_2 & \alpha_2 \end{pmatrix}$ have been analyzed for mean square stability. It is shown that the above stochastic differential equation is mean square stable in the ∞ norm if $\max\{2\lambda_1 + (|\alpha_1| + |\beta_1|)^2, 2\lambda_2 + (|\alpha_2| + |\beta_2|)^2\} < 0$.

(b_1) In [16] for the same stochastic differential equation as in (a) the Mil'stein scheme [11] $X_{n+1} = X_n + \lambda X_n h + \mu X_n \Delta W + \frac{(\Delta W)^2 - h}{2}\mu^2 X_n$ with $O(h^{1.5})$ root mean square error is analyzed for stability in the mean square sense. The mean square stability function $R(h) = |1 + h\lambda|^2 + |h\mu^2| + \frac{1}{2}|h^2\, \mu^4|$ is derived and it is shown that the SDE is stochastically asymptotically stable in the mean square when $R(h) < 1$.

(b_2) The reference [16] also considered the SDE as given in (b) and applied the Euler-Maruyama numerical integration scheme as $X_{n+1} = X_n + hDX_n + BX_n \Delta W$, where h and ΔW stand for step-size and the increment of the Wiener process, respectively. Then the discretized SDE is shown to be stochastically asymptotically stable if

$$\max\{(1+\lambda_1 h)^2 + (|\alpha_1| + |\beta_1|)^2, (1+\lambda_2 h)^2 + (|\alpha_2| + |\beta_2|)^2\} < 1.$$

Next we briefly review a few essential aspects of the logarithmic norm that are used in the stability analysis of ODEs.

(c) In [4,10] the logarithmic norm was introduced in order to derive error bounds for the solution of initial value ODE problems using differential inequalities that distinguish between forward and reverse time integration. This led to requirements for the stability of initial value and boundary value ODE problems. The classical analysis, using the vector norm $\|\cdot\|$ on $\mathbb{C}^n$ and the sub-ordinate matrix norm on $\mathbb{C}^{n\times n}$, defined the logarithmic norm of a matrix A as

$$\mu_p(A) := \lim_{h\to 0^+} \frac{\|I + hA\|_p - 1}{h}.$$

(d) More recently Söderlind [20] considered $f : D \subset X \to X$ and defined least upper bounds (lub) and greatest lower bounds (glb) Lipschitz constants by

$$L[f] = \sup_{u\neq v} \frac{f(u) - f(v)}{|u-v|}, \quad l[f] = \inf_{u\neq v} \frac{f(u)-f(v)}{|u-v|},$$

for u and $v \in D$, where the domain is path connected and

$$l[f]|u-v| \leq |f(u) - f(v)| \leq L[f]|u-v|.$$

If $l[f] > 0$, then $f(u) \to f(v)$ implies that $u \to v$. Then f is an injection, with an inverse on $f(D)$, the same as a matrix $A \in \mathbb{C}^{n\times n}$ being invertible if its glb is strictly positive. Also then, $L[f^{-1}] = l[f^{-1}]$, where $L[f^{-1}]$ is defined over $f(D)$. If $f = A$ is a linear map then $L[A] = \|A\|$. Hence $L[.]$ is left and right G-differentiable for the class of Lipschitz maps. This allows one to define the lub and glb logarithmic Lipschitz constant, by

$$M[f] = \lim_{h\to 0^+} \frac{L[I+hf] - 1}{h}, m[f] = \lim_{h\to 0^-} \frac{L[I+hf]-1}{h}.$$

The lub logarithmic Lipschitz generalizes the classical logarithmic norm for every matrix A, so that, $M[A] = \mu(A)$.

In the following section we develop the stochastic logarithmic norm as an upper bound estimate of the rate of growth of the solution of a multiplicative noise linear SDE. The rate of growth is analyzed as a Dini derivative of the l-th mean in vector p-norm of the solution. This approach may be seen as a special case of the modern definition of logarithmic norm in [20] and as an extension of the classical definition as in [4,10,19]. Other works [8] deal with logarithmic norm of matrix pencils with one invertible matrix. However in our current treatment we do not use matrix pencils for defining the stochastic logarithmic norm.

3. Definition of the Stochastic Logarithmic Norm

A linear Itô SDE with a single channel of multiplicative noise is written as

$$dX_t = AX_t dt + BX_t dW_t, \tag{3}$$

where $A, B \in \mathbb{C}^{n\times n}$, are constant matrices. For the non-linear Itô SDE with a single channel multiplicative noise given by

$$dX_t = f(X,t)dt + g(X,t)dW_t, \tag{4}$$

the strong order 1.0 Itô-Taylor expansion (Mil'stein scheme when used in numerical integration) of X_t at $t = t_{n+1}$ is given as

$$X_{n+1} = X_n + f(X_n,t_n)h + g(X_n,t_n)\Delta W + \left(g\frac{\partial g}{\partial x}\right)_{(X_n,t_n)}\frac{((\Delta W)^2 - h)}{2} + R, \tag{5}$$

where $h = t_{n+1} - t_n$ is the step-size, $W(t_{n+1}) - W(t_n) =: \Delta W \sim N(0,h)$ and R are the $O(h^{1.5})$ remainder terms in the root mean square sense. When linearized at $t = t_n$, $A := \left(\frac{\partial f}{\partial x}\right)_n$ and $B := \left(\frac{\partial g}{\partial x}\right)_n$. We shall define the stochastic logarithmic norm by studying the growth of the l-th mean of the solution X of the linear SDE (3) (linearized SDE in case of SDE (4)) w. r. t. time. We seek an upper bound for the growth rate of $\mathbf{E}(\|X\|_p^l)$ using the upper-right Dini derivative which for any function $\Xi(t)$ w.r.t t is defined as

$$D_+\left(\Xi(t)\right) = \lim_{h\to 0+}\frac{\Xi(t+h) - \Xi(t)}{h}. \tag{6}$$

The strong order 1.0 Itô-Taylor expansion (linearized at t in case of the non-linear SDE (4)) applied to X_t over the time interval $[t, t+h]$ gives

$$X(t+h) = X(t) + hAX(t) + \Delta W BX(t) + \frac{(\Delta W)^2 - h}{2}B^2X(t) + R, \tag{7}$$

with the remainder terms R being $O(h^{1.5})$ in the root mean square sense. In the above expansion (7) $\Delta W := \{\Delta W(t), t \geq 0\}$ are the independent increments of a Wiener process over the interval $[t, t+h]$. Taking the p-norms and raising both sides of (7) to the power of l, we can write the following inequality:

$$\|X(t+h)\|_p^l \leq \|I + hA + \Delta WB + \frac{(\Delta W)^2 - h}{2}B^2 + R_x\|_p^l \|X(t)\|_p^l, \tag{8}$$

where R_x are root mean square $O(h^{1.5})$ remainder terms. For the l-th mean using a p-norm, we apply expectation to both sides of the above inequality and get, almost surely (a.s.),

$$\mathbf{E}\|X(t+h)\|_p^l \leq \mathbf{E}\left(\|I + hA + \Delta WB + \frac{(\Delta W)^2 - h}{2}B^2 + R_x\|_p^l\right)\mathbf{E}\|X(t)\|_p^l. \tag{9}$$

observing that $X(t)$ is independent of the Wiener increment ΔW since the Wiener process is a non-anticipative process. Then, we estimate the expected rate of growth as

$$\mathbf{E}\left(D_+\|X(t)\|_p^l\right) \leq \lim_{h\to 0^+}\frac{\mathbf{E}(\|I + hA + \Delta WB + \frac{(\Delta W)^2 - h}{2}B^2 + R_x\|_p^l) - 1}{h}\mathbf{E}(\|X(t)\|_p^l).$$

Based on the limit term on the right hand side above we introduce the stochastic logarithmic norm.

Definition 2 *The* **stochastic logarithmic norm** *of a square matrix pair of same dimensions, (A, B), in the l-th mean using a matrix p-norm is defined as*

$$\nu_p^l(A,B) = \lim_{h \to 0^+} \frac{\mathbf{E}\left(\|I + hA + \Delta W B + \frac{(\Delta W)^2 - h}{2} B^2\|_p^l \right) - 1}{h}, \tag{10}$$

where the limit is taken in the sense of the existence of the generalized derivative of the Wiener process and $\|A\|_p, \|B\|_p$ are assumed to be finite.

From the above definition the expected rate of growth of the solution can be estimated as $\mathbf{E}\left(D_+\|X(t)\|_p^l\right) \leq \nu_p^l(A,B)\,\mathbf{E}(\|X(t)\|_p^l)$ since R_x are of root mean square $O(h^{1.5})$. Obviously the weaker estimate

$$D_+\mathbf{E}\|X(t)\|_p^l \leq \nu_p^l(A,B)\,\mathbf{E}(\|X(t)\|_p^l)$$

also holds so that $\mathbf{E}\|X(t)\|_p^l \leq e^{\nu_p^l(A,B)t}\|X(t_0)\|_p^l$ for the linear SDE (3). The linear SDE (3) is stochastically stable in the l-th mean using a p-norm when $\nu_p^l(A,B) \leq 0$ and is asymptotically stable in the l-th mean using a p-norm when $\nu_p^l(A,B) < 0$. The linearized SDE (4) (and hence the linear SDE (3)) is numerically stochastically stable in the l-th mean using the p-norm if $\nu_p^l \leq 0$ since over k ($k \ll \infty$) time steps each of size sufficiently small $h_i > 0$ almost surely we have $\mathbf{E}\|X_{t+kh}\|_p^l \leq e^{\sum_{i=1}^k \nu_p^l(A_i,B_i)h}\mathbf{E}\|X_t\|_p^l$ when each $\nu_p^l(A_i,B_i) \leq 0$. If $\mathbf{E}\|X_t\|_p^l$ is finitely bounded w.p. 1, then $\mathbf{E}\|X_{t+kh}\|_p^l$ is almost surely finitely bounded. Later in the paper we relate the stochastic logarithmic norm to the pseudospectrum of the Ito stability matrix $A - \frac{1}{2}B^2$ for the linear SDEs with multiplicative noise.

It may be remarked that for an SDE with additive noise and for an ODE, the stochastic logarithmic norm is given as $\nu_p^1(A,0) = \mu_p(A)$.

4. Some Estimates of the Stochastic Logarithmic Norm

In this section we derive some estimates of the stochastic logarithmic norm using the matrix p-norm, where $p = 1, 2, \infty$. The estimates show the incremental behavior of the stochastic logarithmic norm under perturbations and also the effect of noise on the deterministic ODE.

Let $\lambda_{\max}(A)$ be the largest eigenvalue of any square matrix A. We state the following Lemma from pp.62, [1] which we shall use in estimating the stochastic logarithmic norm while using the matrix 2-norm.

Lemma 1 *Let $A, B \in \mathbb{C}^{n\times n}$ be Hermitian matrices. Then*

$$\lambda_{\max}(A) + \lambda_{\min}(B) \leq \lambda_{\max}(A+B) \leq \lambda_{\max}(A) + \lambda_{\max}(B).$$

The equality holds when $B = kI$, where k is a scalar constant and I is an identity matrix.

For the matrix 2-norm and in the l^{th} mean we can estimate the following from (10).

Theorem 5 *The stochastic logarithmic norm in the l^{th} mean, where $1 \leq l < \infty$ using the matrix 2-norm satisfies the following bounds in the sense of existence of a generalized derivative for the Wiener process as $\zeta dt = dW_t,\ \zeta \sim N(0,1)$.*

$$\nu_2^l(A,B) \le \frac{l}{2}\lambda_{\max}(A+A^H) + \frac{l}{4}\left(\lambda_{\max}(B+B^H)+\lambda_{\max}(-B-B^H)\right)$$
$$+\frac{l}{2}\lambda_{\max}(B^HB) + \frac{l(l-2)}{8}\lambda^2_{\max}(B+B^H), \quad l>2, \tag{11}$$
$$\nu_2^l(A,B) \le \frac{l}{2}\lambda_{\max}(A+A^H) + \frac{l}{4}\left(\lambda_{\max}(B+B^H)+\lambda_{\max}(-B-B^H)\right)$$
$$+\frac{l}{2}\lambda_{\max}(B^HB), \ l \le 2, \tag{12}$$
$$\nu_2^l(A,I) = \frac{l}{2}\lambda_{\max}(A+A^H) + \frac{l}{2} + \frac{l(l-2)}{2}, \tag{13}$$

where $A, B \in \mathbb{R}^{n\times n}$ are the (linearized) drift and diffusion coefficient matrices in a vector SDE with a single channel of Wiener process noise.

Proof: For $1 \le l < \infty$ we have from (10)

$$\nu_2^l(A,B) = \lim_{h\to 0^+} \frac{\mathbf{E}\left(\|I+hA+\Delta WB + \frac{(\Delta W)^2-h}{2}B^2\|_2^l\right)-1}{h} =$$

$$\lim_{h\to 0^+} \frac{\mathbf{E}\left(\sqrt{\lambda_{\max}\left((I+hA+\Delta WB+\frac{(\Delta W)^2-h}{2}B^2)^H(I+hA+\Delta WB+\frac{(\Delta W)^2-h}{2}B^2)\right)}\right)^l}{h}.$$

Then we may write

$$(I+hA+\Delta WB+\frac{(\Delta W)^2-h}{2}B^2)^H(I+hA+\Delta WB+\frac{(\Delta W)^2-h}{2}B^2)$$
$$=(I+hA^H+\Delta WB^H+\frac{(\Delta W)^2-h}{2}(B^H)^2)(I+hA+\Delta WB+\frac{(\Delta W)^2-h}{2}B^2) =$$
$$I+hA+\Delta WB+\frac{(\Delta W)^2-h}{2}B^2+hA^H+\Delta WB^H+(\Delta W)^2B^HB$$
$$+\frac{((\Delta W)^2-h)(B^H)^2}{2}+\cdots = I+h(A+A^H)+\Delta W(B+B^H)+(\Delta W)^2B^HB+$$
$$\frac{(\Delta W)^2-h}{2}(B^2+(B^H)^2)+\cdots.$$

It is possible to write $C = h(A+A^H)+\Delta W(B+B^H)+(\Delta W)^2B^HB+\frac{(\Delta W)^2-h}{2}(B^2+(B^H)^2)$ in a series in the normalized time step size $h \ll 1$. We use the identities $\lambda_{\max}(cA) = |c|\lambda_{\max}(\mathrm{sign}(c)A)$ (where c is a constant, $\mathrm{sign}(c) = \frac{z}{|z|}, z \ne 0$) and $\lambda^2_{\max}(A) = \lambda_{\max}(A^2)$ for estimating the terms in the series. Using the triangle inequality and the identity $[\lambda_{\max}(I+C)]^{\frac{l}{2}} = (1+\lambda_{\max}(C))^{\frac{l}{2}}$, the following can be written:

$$(1+\lambda_{\max}(C))^{\frac{l}{2}} = 1+\frac{l}{2}\lambda_{\max}(C)+\frac{l(l-2)}{8}\lambda_{\max}(C)^2+\ldots = 1+$$
$$\frac{l}{2}\lambda_{\max}\left(h(A+A^H)+\Delta W(B+B^H)+(\Delta W)^2B^HB+\frac{(\Delta W)^2-h}{2}(B^2+(B^H)^2)\right)+$$

$$\frac{l(l-2)}{8}\lambda_{\max}\left(h(A+A^H)+\Delta W(B+B^H)+(\Delta W)^2B^HB+\frac{(\Delta W)^2-h}{2}(B^2+(B^H)^2)\right)^2$$

$$+\cdots\leq 1+\frac{l}{2}h\lambda_{\max}(A+A^H)+\frac{l}{2}\lambda_{\max}(\Delta W(B+B^H))+\frac{l}{2}(\Delta W)^2\lambda_{\max}(B^HB)$$

$$+((\Delta W)^2-h)\frac{l}{4}\lambda_{\max}(B^2+(B^H)^2)+(\Delta W)^2\frac{l(l-2)}{8}\lambda^2_{\max}(B+B^H)+\cdots. \qquad (14)$$

The equality holds when $B=I$ and follows from Lemma 1. Taking expectation on both sides and in the sense of generalized derivative of the Wiener process which is the Gaussian white noise ζ we have $\mathbf{E}(\lambda_{\max}(I+C))^{\frac{l}{2}}=\mathbf{E}(1+\lambda_{\max}(C))^{\frac{l}{2}}\leq 1+\frac{l}{2}h\lambda_{\max}(A+A^H)+\mathbf{E}(|\zeta|)h\frac{l}{4}(\lambda_{\max}(B+B^H)+\lambda_{\max}(-B-B^H))+\mathbf{E}((\Delta W)^2)\frac{l}{2}\lambda_{\max}(B^HB)+(\mathbf{E}((\Delta W)^2)-h)\frac{l}{4}\lambda_{\max}(B^2+(B^H)^2)+\mathbf{E}((\Delta W)^2)\frac{l(l-2)}{8}\lambda^2_{\max}(B+B^H)+O(h^{1.5})=1+h\frac{l}{2}\lambda_{\max}(A+A^H)+h\frac{l}{4}(\lambda_{\max}(B+B^H)+\lambda_{\max}(-B-B^H))+\frac{l}{2}h\lambda_{\max}(B^HB)-h\frac{l(l-2)}{8}\lambda^2_{\max}(B+B^H)+O(h^{1.5})$. For $h\to 0^+$ we obtain

$$\lim_{h\to 0^+}\frac{\mathbf{E}(\lambda_{\max}(1+C))^{\frac{l}{2}}-1}{h}\leq\frac{l}{2}\lambda_{\max}(A+A^H)+\frac{l}{4}(\lambda_{\max}(B+B^H)+\lambda_{\max}(-B-B^H))$$
$$+\frac{l}{2}\lambda_{\max}(B^HB)+\frac{l(l-2)}{8}\lambda^2_{\max}(B+B^H).$$

(b) When $B=I$, we have

$$\lim_{h\to 0^+}\frac{\mathbf{E}(\lambda_{\max}(I+C))^{\frac{l}{2}}-1}{h}=\frac{l}{2}\lambda_{\max}(A+A^H)+\frac{l}{2}\lambda_{\max}(B^HB)+\frac{l(l-2)}{8}\lambda^2_{\max}(B+B^H)$$
$$\Rightarrow\nu_2^l(A,B)=\frac{l}{2}\lambda_{\max}(A+A^H)+\frac{l}{2}+\frac{l(l-2)}{2}.\blacksquare$$

Corollary 1 *For a positive integer $l>2$,*

$$\nu_2^l(A,B)\leq l\left(\mu_2(A)+\frac{1}{2}\|B\|_2^2+\frac{1}{2}(\mu_2(B)+\mu_2(-B))+\frac{l-2}{2}(\mu_2(B))^2\right) \qquad (15)$$

For $l\leq 2$,

$$\nu_2^l(A,B)\leq l\left(\mu_2(A)+\frac{1}{2}(\mu_2(B)+\mu_2(-B))+\frac{1}{2}\|B\|_2^2\right) \qquad (16)$$

Proof: The results (15) and (16) follow directly from (14) in Theorem 5 by Lemma 1 and the Itô isometry for the expectation of the Itô integrals.■

The inequality (15) has been used for mean square stability estimates in [17].

Corollary 2 *If $l=1,2$ and $p=2$, that is, in the mean and in the mean square and using the matrix two norm, the following results hold when B is the identity matrix I.*

$$\nu_2^1(A,I) = \lambda_{\max}\left(\frac{A+A^H}{2}\right) + \frac{1}{2} - \frac{4}{8} = \lambda_{\max}\left(\frac{A+A^H}{2}\right) = \mu_2(A), \tag{17}$$

$$\nu_2^2(A,I) = \lambda_{\max}\left(A+A^H\right) + 1 = 2\mu_2(A) + 1, \tag{18}$$

Proof: The results follow from Theorem 5, by substituting $p = 2, l = 1$ and $p = 2, l = 2$.■

Some properties of the stochastic logarithmic norm under perturbation is obtained in the following result. In the generalized sense of the Wiener process derivative a couple of useful inequalities showing how the Wiener process perturbs the deterministic ODE is also given.

Theorem 6 *For any square matrices of same dimensions, $B, A, \Delta A, \Delta B$ and a real number $\alpha > 0$ the stochastic logarithmic norm has the following properties.*

$$\nu_p^l(\alpha A, \sqrt{\alpha}B) = \alpha\nu_p^l(A,B) \tag{19}$$

$$\nu_p^1(A+\Delta A, B+\Delta B) \leq \nu_p^1(A, \sqrt{2}B) + \nu_p^1(\Delta A, \sqrt{2}\Delta B) + \frac{1}{\sqrt{2}}\|(B-\Delta B)^2\|_p \tag{20}$$

$$\nu_p^l(A+\Delta A, B+\Delta B) \leq \nu_p^l(A, \frac{B+\Delta B}{\sqrt{2}}) + \nu_p^l(\Delta A, \frac{B+\Delta B}{\sqrt{2}}) \tag{21}$$

$$\nu_p^l(A,0) = l\mu_p(A) \tag{22}$$

Further, in the sense of the existence of a generalized derivative of the Wiener process which is a Gaussian white noise, the following estimate holds for a positive integer l and for any of the matrix p-norms.

$$\nu_p^l(A,B) \leq l\mu_p(A) + \frac{l}{2}\mu_p(-B^2) + \frac{l(l+1)}{4}\|B\|_p^2 + l\|B\|_p \tag{23}$$

$$\leq l\mu_p(A) + l\|B\|_p\left(1 + \frac{l+3}{4}\|B\|_p\right) \leq l\mu_p(A) + l\left(1 + \frac{l+3}{4}\|B\|_p\right)^2 \tag{24}$$

Proof: We can scale the Wiener process as $\sqrt{\alpha}$ and write

$$\begin{aligned}\nu_p^l(\alpha A, \sqrt{\alpha}B) &= \lim_{h\to 0^+} \alpha\frac{\mathbf{E}\left(\|I + (\alpha h)A + \Delta(\sqrt{\alpha}W)B + \frac{(\Delta\sqrt{\alpha}W)^2 - \alpha h}{2}B^2\|_p^l\right) - 1}{\alpha h} \\ &= \alpha\nu_p^l(A,B)\end{aligned}$$

for any positive integer l and any of the matrix p-norms.

From the definition of the stochastic logarithmic norm and scaling the Wiener process, we can write

$$\nu_p^1(A+\Delta A, B+\Delta B) \leq \lim_{h\to 0^+}\left(\frac{\mathbf{E}(\|I + (2h)A + (\sqrt{2}\Delta W)(\sqrt{2}B) + \frac{(\Delta\sqrt{2}W)^2 - 2h}{2}(\sqrt{2}B)^2\|_p) - 1}{2h}\right.$$
$$+\frac{\mathbf{E}(\|I + (2h)\Delta A + (\sqrt{2}\Delta W)(\sqrt{2}\Delta B) + \frac{(\Delta\sqrt{2}W)^2 - 2h}{2}(\sqrt{2}\Delta B)^2\|_p) - 1}{2h} +$$
$$\left.\frac{\mathbf{E}|\int_0^h W_u dW_u|}{h}\|B\Delta B + \Delta B B - B^2 - (\Delta B)^2\|_p\right) \leq \nu_p^1(A, \sqrt{2}B) + \nu_p^1(\Delta A, \sqrt{2}\Delta B) + \frac{1}{\sqrt{2}}\|(B-\Delta B)^2\|_p$$

since $\lim_{h\to 0^+} \frac{\mathbf{E}|\int_0^h W_u dW_u|}{h} = h/(\sqrt{2}h) = \frac{1}{\sqrt{2}}$. For any positive integer l, the above can be re-written as

$$\begin{aligned}
\nu_p^l(A+\Delta A, B+\Delta B) \le \lim_{h\to 0^+} \Bigg(\Bigg(& 2^{l-1}\frac{1}{2^{l-1}}\mathbf{E}\|I+(2h)A+(\sqrt{2}\Delta W)(\sqrt{2}\frac{B+\Delta B}{2}) \\
& + \frac{(\Delta\sqrt{2}W)^2-2h}{2}(\sqrt{2}\frac{B+\Delta B}{2})^2\|_p^l - 1\Bigg)/(2h) \\
& + \Bigg(2^{l-1}\frac{1}{2^{l-1}}\mathbf{E}\|I+(2h)\Delta A+(\sqrt{2}\Delta W)(\sqrt{2}\frac{B+\Delta B}{2}) \\
& + \frac{(\Delta\sqrt{2}W)^2-2h}{2}(\sqrt{2}\frac{B+\Delta B}{2})^2\|_p^l - 1\Bigg)/(2h)\Bigg) \\
& \le \nu_p^l(A, \frac{B+\Delta B}{\sqrt{2}}) + \nu_p^l(\Delta A, \frac{B+\Delta B}{\sqrt{2}}).
\end{aligned}$$

Applying (20) recursively to $\nu_p^l(A+0, B/2+B/2)$ and using (19), we obtain $\nu_p^l(A,B) \le \lim_{n\to\infty}(\nu_p^l(A, \frac{1}{(\sqrt{2})^n}B) + (2^n-1)\nu_p^l(0, \frac{1}{(\sqrt{2})^n}B)) = \nu_p^l(A,0) + \lim_{n\to\infty}(2^n/2^n)\nu_p^l(0,B) + 0 = \nu_p^l(A,0)+\nu_p^l(0,B)$. For $l=1$, the inequality reduces to $\nu_p^1(A,B) \le \mu_p(A)+\nu_p^1(0,B)$.

For the deterministic case with no noise, we have

$$\begin{aligned}
\nu_p^l(A,0) &= \lim_{h\to 0^+} \frac{\|I+hA\|_p^l - 1}{h} \\
&= \lim_{h\to 0^+} l\|I+hA\|_p^{l-1} D_{+,h}\|I+hA\|_p \\
&= \lim_{\epsilon\to 0^+} l\frac{\|I+\epsilon A\|_p - 1}{\epsilon} \\
&= l\mu_p(A).
\end{aligned}$$

For the stochastic logarithmic norm of the diffusion coefficient B it is possible to write

$$\begin{aligned}
\nu_p^l(0,B) &= \lim_{h\to 0^+} \frac{\mathbf{E}\|I+\Delta W B + \frac{(\Delta W)^2-h}{2}B^2\|_p^l - 1}{h} \\
&\le \lim_{h\to 0^+} \left(\frac{\|I-(2h)\frac{1}{2}B^2\|_p^l - 1}{(2h)} + \frac{\mathbf{E}\|I+2B\Delta W + B^2(\Delta W)^2\|_p^l - 1}{(2h)} \right) \\
&\le \nu_p^l(-\frac{1}{2}B^2, 0) + \frac{l}{2}\|B\|_p^2 + l\|B\|_p + \frac{l(l-1)}{4}\|B\|_p^2
\end{aligned}$$

in the sense of the existence of a generalized derivative of the Wiener process so that $\int_0^h \xi dt = \int_0^h dW_s$ where $\xi \sim N(0,1)$ is a Gaussian white noise. Further we have $\nu_p^l(-\frac{1}{2}B^2, 0)$ $= \frac{l}{2}\mu_p(-B^2)$ and $\frac{l}{2}|\mu_p(-B^2)| \le \frac{l}{2}\|B\|_p^2$ from the properties of the logarithmic norm [5]. Hence the estimates (23) and (24) . ■

We remark that the stochastic logarithmic norm does not satisfy the triangle inequality property nor the that of the multiplication by a scalar in the same way as the (deterministic) logarithmic norm. However, it is consistent with Itô calculus in its property (19) of multiplication with a scalar. Again, Itô calculus makes the noise "redistribute" for any

additive perturbation to the drift and diffusion coefficients as found in (21).

The logarithmic norm coincides with the stochastic logarithmic norm in special cases. One such case is obtained from Theorem 5 by setting $l = 1$ and $B = I$ (and also by setting $B = 0$ in the trivial case) for $p = 2$.

Theorem 7 $\max_\lambda \Re(\lambda(A)) \leq \frac{1}{2}\nu_2^2(A, I) - \frac{1}{2}$ *where I is the identity matrix.*

Proof: From (18) we have $\nu_2^2(A, I) = 2\mu_2(A) + 1$. The logarithmic norm has the lower bound property [5]: $\max_\lambda \Re(\lambda(A)) \leq \mu_2(A)$. Hence the inequality. ■

8. Conditions for Mean and Mean Square Stability

We recall that the linear SDE (3) is stochastically stable in the l-th mean using a p-norm if $\nu_p^l(A, B) \leq 0$. To estimate of the stability of a linear SDE one of the upper bounds derived in the last section may be used, especially, when it is needed to compute incrementally the effect of adding noise to an ODE or an SDE with known stability estimates. In practice, when using the upper bounds for estimating the stability of an SDE incrementally, a small positive number (determined by the stochastic stability region of the numerical integrator) is used as cut-off rather than a very small tolerance or zero so that the effect of non-normality and stiffness [6] in the SDE's (both linear and locally linearized) transient numerical behavior in the stochastic logarithmic norm is taken into account.

The following results are a couple of special cases of the stochastic logarithmic norm approach to the stability of SDEs with multiplicative noise.

- We consider the scalar case $A = \alpha \in \mathbb{C}$ and $B = \beta \in \mathbb{C}$ in (3). Then we apply Theorem 5(a) and consider (15) for $l = 1, 2$ the condition $\nu_p^l(A, B) < 0$, so that for $p = 2$, we get the condition for stochastic stability in the mean as

$$\Re(\alpha) + \frac{1}{2}|\beta|^2 \leq 0 \tag{25}$$

and in the mean square as

$$2\Re(\alpha) + |\beta|^2 \leq 0. \tag{26}$$

The above conditions are essentially the same but in practice the 0 on the right hand side is replaced by TOL which is a small positive real number [13] and thus l becomes significant for estimates in higher moments. These results are the same as in review (a) of Section 2.

- For the matrices $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and $B = \begin{pmatrix} \alpha_1 & \beta_1 \\ \beta_2 & \alpha_2 \end{pmatrix}$, the SDE (3) is mean square stable in the ∞-norm if

$$2\max\{\lambda_1, \lambda_2\} + 2\left(\frac{5}{4}\max\{|\alpha_1| + |\beta_1|, |\alpha_2| + |\beta_2|\} + 1\right)^2 \leq 0. \tag{27}$$

The above condition is obtained using (24) in which

$$\nu_\infty^2(A, 0) = \lim_{h \to 0^+} \frac{(\max\{|1 + h\lambda_1|, |1 + h\lambda_2|\})^2 - 1}{h}$$

$$
\begin{aligned}
&= \lim_{h\to 0^+} 2\max\{|1+h\lambda_1|, |1+h\lambda_2|\}\max\{\lambda_1,\lambda_2\} \\
&= 2\max_{i=1,2}\lambda_i
\end{aligned}
$$

and $\|B\|_\infty = \max\{|\alpha_1|+|\beta_1|, |\alpha_2|+|\beta_2|\}$.

9. Direct Computation of the Stochastic Logarithmic Norm and Sharper Bounds

Theorem 10 *In the sense of the existence of a generalized derivative for the Wiener process, i.e., $\zeta dt = dW_t,\ \zeta \sim N(0,1)$, we can compute the stochastic logarithmic norm directly as follows.*

$$\nu_p^l(A,B) = l\mathbf{E}\left(\mu_p\left(A - \frac{1}{2}B^2 + B\zeta\right)\right)$$

Proof:

$$
\begin{aligned}
\nu_p^l(A,B) &= \lim_{h\to 0^+} \frac{\mathbf{E}\|I + hA - \frac{1}{2}B^2 h + B\Delta W + \frac{1}{2}B^2(\Delta W)^2\|_p^l - 1}{h} \\
&= \lim_{h\to 0^+} \mathbf{E}\Big(l\|I + hA - \frac{1}{2}B^2 h + B\Delta W + \frac{1}{2}B^2(\Delta W)^2\|_p^{l-1} \\
&\quad \times D_{+,h}\|I + hA - \frac{1}{2}B^2 h + B\Delta W + \frac{1}{2}B^2(\Delta W)^2\|_p \Big) \\
&= l \lim_{\epsilon\to 0^+} \frac{\mathbf{E}\|I + (A - \frac{1}{2}B^2)\epsilon + B\int_0^\epsilon \zeta ds + \frac{1}{2}B^2\left(\int_0^\epsilon \zeta ds\right)^2\| - 1}{\epsilon} \\
&= l\mathbf{E}\left(\mu_p\left(A - \frac{1}{2}B^2 + B\zeta\right)\right), \qquad (28)
\end{aligned}
$$

where the last equality follows from considering the Itô formula along with the generalized derivative of the Wiener process. ■

The above result shows that the stochastic logarithmic norm is the expected logarithmic norm behavior of the SDE. Consequently we can derive the following inequalities.

Corollary 3

$$\nu_p^l(A,B) \le l\mu_p(A) + \frac{l}{2}\left(\mu_p(-B^2) + \mu_p(B) + \mu_p(-B)\right) \qquad (29)$$

$$\nu_p^l(A,B) \ge l\mu_p(A) - \frac{l}{2}\left(\mu_p(B^2) + \mu_p(B) + \mu_p(-B)\right) \qquad (30)$$

Proof: From (28) we have $\nu_p^l(A,B) \le l\mu_p(A) + \frac{l}{2}\left(\mu_p(-B^2) + \mathbf{E}(\mu_p(B\zeta))\right)$ in view of the triangular inequality of the logarithmic norm. Since $\zeta \sim N(0,1)$, we have $\mathbf{E}(\mu_p(B\zeta)) = \mathbf{E}|\zeta|(\mu_p(B) + \mu_p(-B))/2 = (\mu_p(B) + \mu_p(-B))/2$. Again, $\mu_p(A) = \mu_p(A - \frac{1}{2}B^2 + B\zeta + \frac{1}{2}B^2 - B\zeta) \le \mu_p(A - \frac{1}{2}B^2 + B\zeta) + \mu_p(\frac{1}{2}B^2) + \mu_p(-B\zeta)$ so that $l\mu_p(A) - l\mu_p(\frac{1}{2}B^2) - l\mathbf{E}\mu_p(-B\zeta) \le \nu_p^l(A,B)$. Hence the inequalities. ■

Corollary 4 *The stochastic logarithmic norm can be bounded as follows.*

$$\left|\nu_p^l(A,B)\right| \le l\left|\mu_p(A-\frac{1}{2}B^2)\right| + l\mathbf{E}\left|\mu_p(B\zeta)\right| \le l\left\|A-\frac{1}{2}B^2\right\|_p + l\left\|B\right\|_p \tag{31}$$

Proof: By Theorem 10 we have $\left|\nu_p^l(A,B)\right| = l\left|\mathbf{E}\left(\mu_p\left(A-\frac{1}{2}B^2+B\zeta\right)\right)\right|$. Then, by Jensen inequality one obtains, a.s.,

$$l\left|\mathbf{E}\left(\mu_p\left(A-\frac{1}{2}B^2+B\zeta\right)\right)\right| \le l\mathbf{E}\left|\left(\mu_p\left(A-\frac{1}{2}B^2+B\zeta\right)\right)\right|.$$

From the triangular inequality we have

$$\left|\mu_p\left(A-\frac{1}{2}B^2+B\zeta\right)\right| \le \left|\mu_p\left(A-\frac{1}{2}B^2\right)\right| + \left|\mu_p(B\zeta)\right|$$

and the bound property of the logarithmic norm leads to

$$\left|\nu_p^l(A,B)\right| \le l\left\|A-\frac{1}{2}B^2\right\|_p + l\mathbf{E}\left\|B\zeta\right\|_p \le l\left\|A-\frac{1}{2}B^2\right\|_p + l\left\|B\right\|_p . \blacksquare$$

11. Examples

11.1. *Stabilization of an inverted pendulum*

It is well known [18,14] that a vertical inverted pendulum can be stabilized around a mean vertical position by application of a suitable highly oscillatory excitation in the form of an appropriate noise. We compute the stochastic logarithmic norm of such a system to show how an appropriate noise stabilizes the system. The equation of the inverted pendulum may be written as

$$d\theta = v\ dt + \epsilon v\ dW_t,\ 1 \gg \epsilon > 0 \tag{32}$$

$$dv = \frac{g}{l}\theta\ dt + b\theta\ dW_t, \tag{33}$$

where g is acceleration due to gravity, $l > 0$ is the effective length of the pendulum and θ is the small angular displacement from the mean verticalposition, i.e, $\theta = 0$, so that $A = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & \epsilon \\ b & 0 \end{pmatrix}$. It may be noted that $\mu_2(A) = \frac{1}{2} + \frac{g}{2l}$ and $\Re\lambda_{\max}(A) = \frac{g}{l} > 0$. Thus the system without the Wiener process excitation is unstable. The stochastic logarithmic norm of the Wiener process excited system may be computed as

$$\nu_2^2(A,B) = \mathbf{E}\left(\max\left\{1+\frac{g}{l}+(b+\epsilon)\zeta - b\epsilon, -\left(1+\frac{g}{l}+(b+\epsilon)\zeta + b\epsilon\right)\right\}\right),$$

where $\zeta \sim N(0,1)$. For stabilizing the pendulum around the mean position $\theta = 0$ we require $\nu_2^2(A,B) \le 0$ and get the condition that

$$b \ge \frac{1}{\epsilon}\left(1+\frac{g}{l}\right) \tag{34}$$

in which $\epsilon/(2(1+g/l))$ can be interpreted physically as the amplitude of a very wide band vertical excitation (as an approximation to Wiener process) at the base of the pendulum. Since a Karhunen-Loeve expansion of the Wiener process [9] contains the high frequency terms, this also shows how the above result is consistent with the result (in [14]) that

a small amplitude highly oscillatory wide band vertical excitation stabilizes a vertical inverted pendulum.

11.2. *Nonnormality*

From [7] we take this linear SDE in the form of (3) in which $A := \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$, $B := \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}$, $b \in \mathbb{R}$. Obviously the system without any Wiener process excitation, i.e, the deterministic ODE is asymptotically stable but $\mu_2(A) = \max\{\frac{b}{2} - 1, -\left(\frac{b}{2} + 1\right)\}$ due to non-normality of A and the ODE system tends to be numerically unstable in its transient behavior when $|b| > 2$. Formulating the system as a multiplicative noise SDE we compute the stochastic logarithmic norm directly from (28) in the mean square and using the 2-norm as in [7]:

$$\nu_2^2 = \max\{\sigma^2 - 2 \pm b\}$$

whence it is required that $\sigma^2 \leq \min\{2 \mp b\}$ so that $\nu_2^2 \leq 0$ for the mean square stability of the SDE. If $\sigma \in \mathbb{R}$ and $|b| > 2$, then it is not possible to numerically stabilize the SDE in the mean square by choosing an appropriate σ. For $|b| > 2$ and $\sigma \in \mathbb{C}$, $\sigma^2 \leq 2 - |b|$ would be sufficient for exploiting the noise towards numerically stabilizing the SDE system in the mean square. If $\sigma, b \in \mathbb{R}$ and $\sigma := b^{-\frac{1}{4}}$, then the SDE system is stochastically stable when $1 \geq b \geq \frac{3-\sqrt{5}}{2}$.

11.3. *Numerical examples*

Example 1 *We consider*

(*a*) $A = \begin{pmatrix} -100 & 0 \\ 0 & -200 \end{pmatrix}, B = \begin{pmatrix} 5 & 0 \\ 0 & 6 \end{pmatrix}$.

(*b*) $A = \begin{pmatrix} -100 & 0 \\ 200 & -200 \end{pmatrix}, B = \begin{pmatrix} 5 & 2 \\ 0 & 6 \end{pmatrix}$.

(*c*) $A = \begin{pmatrix} -100 & 20 \\ 0 & -200 \end{pmatrix}, B = \begin{pmatrix} 5 & 2 \\ 0 & 6 \end{pmatrix}$.

(*d*) $A = \begin{pmatrix} -100 + 20i & 0 \\ 2 & -200 + i \end{pmatrix}, B = \begin{pmatrix} 5 + i & 0 \\ 2i & -6 - 10i \end{pmatrix}$.

(*e*) $A = \begin{pmatrix} -100 & 20 \\ 7 & -200 \end{pmatrix}, B = \begin{pmatrix} 5 & 2 \\ 4 & 6 \end{pmatrix}$.

(*f*) $A = -100, B = 10$ *([16])*.

(*g*) $A := \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix}; B := \begin{pmatrix} B_1 & B_{12} \\ 0 & B_2 \end{pmatrix}$ *where* $A_1 := \begin{pmatrix} 0.1 & 4 & 20 \\ 0 & 0.1 & 5 \\ 0 & 0 & 0.1 \end{pmatrix}$,

$A_2 := \begin{pmatrix} -0.2 & 3 & 100 \\ 0 & -0.2 & 50 \\ 0 & 0 & -0.2 \end{pmatrix}, B_1 := \begin{pmatrix} 2 & 30 & 10 \\ 0 & 2 & 50 \\ 0 & 0 & 2 \end{pmatrix}, B_2 := \begin{pmatrix} 4 & 6 & 20 \\ 0 & 4 & 40 \\ 0 & 0 & 4 \end{pmatrix}$,

$A_{12} = \begin{pmatrix} 2.2857 \times 10^{-2} & -2.3547 \times 10^{-2} & -6.8279 \times 10^{-2} \\ 9.3914 \times 10^{-2} & -9.6719 \times 10^{-2} & -2.8049 \times 10^{-1} \\ 2.8585 \times 10^{-1} & -2.9443 \times 10^{-1} & -8.5382 \times 10^{-1} \end{pmatrix}$,

$$\text{and } B_{12} = \begin{pmatrix} 1.2606 \times 10^{-1} & -4.6007 \times 10^{-1} & 7.0963 \times 10^{-3} \\ 1.8156 \times 10^{-1} & -6.6259 \times 10^{-1} & 1.0235 \times 10^{-2} \\ 1.4481 \times 10^{-1} & -5.2845 \times 10^{-1} & 8.1625 \times 10^{-3} \end{pmatrix}.$$

(h) $A = 100 \times C_{100\times100}, B = 100 \times D_{100\times100};\ C_{ij}, D_{ij} \sim U(0,1)$.

(i) $\begin{pmatrix} -100 & 0 \\ 0 & -1 \end{pmatrix}$. $B = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ [17]

Example	Lbound	$\nu_2^2(A,B)$	Ubound
(a)	-1.1239×10^2	-1.0470×10^2	-4.0393×10^1
(b)	-1.1919×10^2	-1.1468×10^2	-3.1393×10^1
(c)	-2.4082×10^2	-2.2415×10^2	-1.5302×10^2
(d)	-2.2490×10^2	-2.2354×10^2	-5.9075×10^1
(e)	-2.6837×10^2	-2.3232×10^2	-1.21915×10^2
(f)	-3.0000×10^2	-3.0026×10^2	-1.0000×10^2
(g)	-9.1852×10^2	9.2453×10^2	4.8398×10^3
(h)	-2.5191×10^7	1.2369×10^5	2.5330×10^7
(i)	-6.0000	-5.91409	-2.0000

Table 1
The table gives values $\nu_2^2(A,B)$ and compares them with the estimates *Ubound* and *Lbound*.

In Table 1 Ubound is computed using the right hand side in (29) with $p = 2,\ l = 2$ and Lbound is computed using the right hand side in (30). The stochastic logarithmic norm is computed using (28).

12. Extension to Multiple Noise Channels

The definition of the stochastic logarithmic norm can be extended to the vector SDE with multiple channels of multiplicative noise (i.e., with multi-dimensional Wiener process). Considering the Itô-Taylor strong order 1.0 (Mil'stein) scheme for the SDE

$$dX_t = AX_t dt + \sum_{j=1}^{m} B^{(j)} X_t dW_t^{(j)}, \tag{35}$$

where each $W^{(j)}$ is an independent component Wiener process, we can define the following.

Definition 3 *The* **stochastic logarithmic norm** *for a tuple of $m+1$ square matrices of same dimensions, $(A, B^{(1)}, B^{(2)}, \cdots, B^{(m)})$, i.e., $\left(A, B^{(1:m)}\right)$, which are the (linearized) drift and diffusion coefficients of a (non-linear) vector SDE with m channels of multiplicative noise, is defined as*

$$\nu_p^l\left(A, B^{(1:m)}\right) = \lim_{h\to 0^+} \frac{\mathbf{E}\|I + hA + \sum_{j=1}^{m} B^{(j)}\Delta W^{(j)} + \sum_{i=1}^{m}\sum_{j=1}^{m} B^{(i)}B^{(j)}\int_0^h\int_0^s dW_u^{(i)}dW_s^{(j)}\|_p^l - 1}{h},$$

where the limit is taken in the sense of the existence of the generalized derivative of the Wiener process and $\|A\|_p$ *and each* $\|B^{(i)}\|_p$ *are assumed to be finite.*
For $p = 2, l \geq 2$ it is easy to obtain the following estimate after proceeding as in the estimate in (15).

$$\nu_2^l(A, B^{(1:m)}) \leq l\mu_2(A) + \frac{l}{2}\sum_{j=1}^{m}\|B^{(j)}\|_2^2 + \frac{l}{2}\sum_{j=1}^{m}(\mu_2(B^{(j)}) + \mu_2(-B^{(j)})) + \frac{l(l-2)}{2}\sum_{i=1}^{m}(\mu_2(B^{(i)}))^2 \tag{36}$$

In general, for any p, it is possible to estimate $\nu_p^l(0, B^{(1:m)})$ in the sense of the existence of a generalized derivative of the Wiener process such that $dW^{(i)} = \zeta^{(i)}dt,\ \zeta^{(i)} \sim N(0,1),\ \mathbf{E}(\zeta^{(i)}\zeta^{(j)}) = 0$ for $i \neq j$:

$$\begin{aligned}
\nu_p^l(0, B^{(1:m)}) &= \lim_{h\to 0^+} \frac{\mathbf{E}\|I + \sum_{j=1}^{m} B^{(j)}\Delta W^{(j)} + \sum_{i=1}^{m}\sum_{k=1}^{m} B^{(i)}B^{(k)}\int_0^h\int_0^s dW_u^{(i)}dW_s^{(k)}\|_p^l - 1}{h} \\
&\leq \lim_{h\to 0^+}\Bigg(\frac{\|I - (2h)\sum_{i=1}^{m}\frac{1}{2}{B^{(i)}}^2\|_p^l - 1}{(2h)} + \\
&\frac{\mathbf{E}\|I + 2\Delta W\sum_{i=1}^{m} B^{(i)} + (\Delta W)^2\sum_{i=1}^{m}{B^{(i)}}^2 + 2\sum_{i=1}^{m}\sum_{j=1,\ i\neq j}^{m} B^{(i)}B^{(j)}\int_0^h\int_0^s dW_u^{(i)}dW_s^{(j)}\|_p^l - 1}{(2h)}\Bigg) \\
&\leq \frac{l}{2}\mu_p(-\sum_{i=1}^{m} B^{(i)}) + l\sum_{i=1}^{m}\|B^{(i)}\|_p + \frac{l}{2}\sum_{i=1}^{m}\|B^{(i)}\|_p^2 + \frac{l}{\sqrt{2}}\sum_{i=1}^{m}\sum_{j=1,\ i\neq j}^{m}\|B^{(i)}B^{(j)}\|_p.
\end{aligned}$$

Then we can upper bound the stochastic logarithmic norm as

$$\nu_p^l(A, B^{(1:m)}) \leq l\mu_p(A) - \frac{l}{2}\mu_p(\sum_{i=1}^{m} B^{(i)}) + l\sum_{i=1}^{m}\|B^{(i)}\|_p + \frac{l}{2}\sum_{i=1}^{m}\|B^{(i)}\|_p^2 + \frac{l}{\sqrt{2}}\sum_{i=1}^{m}\sum_{j=1,\ i\neq j}^{m}\|B^{(i)}B^{(j)}\|_p. \tag{37}$$

Similar to (28) we can compute the stochastic logarithmic norm for the multi-channel case with

$$\nu_p^l(A, B^{(1:m)}) = l\mathbf{E}\left(\mu_p\left(A - \frac{1}{2}\sum_{i=1}^{m}{B^{(i)}}^2 + \sum_{i=1}^{m} B^{(i)}\zeta^{(i)}\right)\right), \tag{38}$$

and bound it as in (31) as

$$\left|\nu_p^l(A, B^{(1:m)}\right| \leq l \left\| A - \frac{1}{2}\sum_{i=1}^m {B^{(i)}}^2 \right\|_p + l \sum_{i=1}^m \left\| B^{(i)} \right\|_p . \tag{39}$$

From (38) and as in Corollary 3, the stochastic logarithmic norm for the multiplicative multiple channel noise can be bounded as

$$\begin{aligned} l\mu_p(A) - \frac{l}{2}\sum_{i=1}^m \left(\mu_p({B^{(i)}}^2) + \mu_p(B^{(i)}) + \mu_p(-B^{(i)})\right) &\leq \nu_p^l(A, B^{(1:m)}) \\ \leq l\mu_p(A) + \frac{l}{2}\sum_{i=1}^m \left(\mu_p(-{B^{(i)}}^2) + \mu_p(B^{(i)}) + \mu_p(-B^{(i)})\right) . \end{aligned} \tag{40}$$

13. Relationship with Pseudospectrum

We have mentioned in the introduction that the logarithmic norm as a bound on the pseudospectrum of the stability matrix provides an estimate of the finite time interval numerical stability of an ODE. The finite time interval numerical stability differs from the asymptotic stability in capturing the effect of nonnormality of the stability matrices and local stiffness that affect the computation of the numerical integration. In SDEs with multiplicative noise the diffusion coefficients may significantly affect this transient numerical stability of an SDE. It may be recalled that balanced methods have been designed [12] to overcome difficulties arising from stiffness in both drift and diffusion. The stochastic logarithmic norm relates to the pseudospectrum of the drift coefficients by way of diffusion coefficients acting as perturbations and thus captures the expected transient stability of the SDE. The estimate of stability based on the stochastic logarithmic norm can then be used for selecting an appropriate stiff stochastic integrator.

Let $\| -\frac{1}{2}\sum_{i=1}^m {B^{(i)}}^2 + \sum_{i=1}^m B^{(i)}\zeta^{(i)}\|_2 = \beta$. From the definition of pseudospectrum [15], we may write:

$$\begin{aligned} \mathbf{E}\max_\lambda \Re\lambda_\beta(A) &= \mathbf{E}\max_\lambda \Re\lambda\left(A - \frac{1}{2}\sum_{i=1}^m {B^{(i)}}^2 + \sum_{i=1}^m B^{(i)}\zeta^{(i)}\right) \\ &\leq \mathbf{E}\left(\mu_2\left(A - \frac{1}{2}\sum_{i=1}^m {B^{(i)}}^2 + \sum_{i=1}^m B^{(i)}\zeta^{(i)}\right)\right) \\ &= \frac{1}{2}\nu_2^2(A, B^{(1:m)}) \end{aligned} \tag{41}$$

using (38). For small noise with $1 \gg \beta > 0$, obviously, the stochastic logarithmic norm gives an upper bound estimate of the mean transient numerical stability behavior of the deterministic ODE $dX_t = AX_t dt$. Denoting $\gamma = \|\sum_{i=1}^m B^{(i)}\zeta^{(i)}\|_2$, and proceeding in a similar fashion as in (41) one obtains

$$\mathbf{E}\max_\lambda \Re\lambda_\gamma\left(A - \frac{1}{2}\sum_{i=1}^m {B^{(i)}}^2\right) \leq \mathbf{E}\left(\mu_2\left(A - \frac{1}{2}\sum_{i=1}^m {B^{(i)}}^2 + \sum_{i=1}^m B^{(i)}\zeta^{(i)}\right)\right)$$

$$= \frac{1}{2}\nu_2^2(A, B^{(1:m)}). \tag{42}$$

In the above inequalities the stochastic logarithmic norm appears as an upper bound on the mean maximum real part of the perturbed spectrum of $A - \frac{1}{2}\sum_{i=1}^{m} {B^{(i)}}^2$ which is significant for the stochastic asymptotic stability of the SDE (35).

14. Conclusion

This paper has extended the classical logarithmic norm to define the stochastic logarithmic norm for the numerical stability analysis of vector Itô stochastic differential equations with multi-dimensional multiplicative noise. Incremental estimates of the stochastic logarithmic norm due to perturbations and bounds with respect to logarithmic norm of the drift and diffusion coefficient matrices have been studied. Further investigation relating the stochastic logarithmic norm to the pseudo-spectrum and stiffness, both in drift and diffusion, is needed since the stochastic logarithmic norm gives an upper bound on the mean maximum real part of the pseudospectrum of a matrix perturbed by the noise. This last property may be used in choosing stiff and balanced numerical integrators and a detailed study for various class of integrators in this respect remains to be done. Detailed study of application of the stochastic logarithmic norm to the stability analysis of non-linear SDE is necessary too.

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